Some bad bounds on ω and related sums

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1 Definition

We use p_n to denote the nth prime number. We define $\omega(p_n)$ as the smallest integer with the properties

$$\omega(p_n) > p_{n+1} - p_n \tag{1}$$

$$\omega(p_n) | p_n - (p_{n+1} - p_n) \tag{2}$$

If $\omega(p_n)$ does not exist, we define it as 0. Furthermore we define

$$\zeta(p_n) = \frac{p_n - (p_{n+1} - p_n)}{\omega(p_n)} \tag{3}$$

We will call $\omega(p_n)$ the weight of the prime p_n while we use the term level for $\zeta(p_n)$. It can be shown that for all $n \geq 5$

$$2 < \omega(p_n) < p_n - 6 \tag{4}$$

2 Basic theorems

Theorem We have

$$\lim_{n\to\infty} \frac{\sum_{i=5}^{n} \omega(p_i)\zeta(p_i)}{n^2 \ln(n)} = \frac{1}{2} \tag{5}$$

Proof We will use the statement $2p_n - p_{n+1} = \omega(p_n)\zeta(p_n)$ to start.

$$\sum_{i=5}^{n} \omega(p_i)\zeta(p_i) = \sum_{i=5}^{n} p_i - (p_{i+1} - p_i) = \sum_{i=5}^{n} p_i - \sum_{i=5}^{n} (p_{i+1} - p_i) \tag{6}$$

Due to a theorem of Bach and Shallit we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} p_n}{n^2 \ln(n)} = \frac{1}{2} \tag{7}$$

Therefore we see that

$$\frac{\sum_{i=5}^{n}\omega(p_i)\zeta(p_i)}{n^2\ln(n)} = \frac{\sum_{i=5}^{n}p_n}{n^2\ln(n)} - \frac{\sum_{i=5}^{n}(p_{i+1}-p_i)}{n^2\ln(n)} \tag{8}$$

Now it is easy to understand that

$$\lim_{n\to\infty}\frac{\sum_{i=5}^{n}\omega(p_i)\zeta(p_i)}{n^2\ln(n)} = \lim_{n\to\infty}\frac{\sum_{i=5}^{n}p_n}{n^2\ln(n)} - \frac{\sum_{i=5}^{n}(p_{i+1}-p_i)}{n^2\ln(n)} = \frac{1}{2} - \lim_{n\to\infty}\frac{\sum_{i=5}^{n}(p_{i+1}-p_i)}{n^2\ln(n)} \tag{9}$$

The last expression is nothing more than a telescopic sum having the value $\frac{p_{n+1}-11}{n^2\ln(n)}$ which tends against 0 rather quickly.

3 $\omega(p_n)$ and $\zeta(p_n)$ in Zone 2/3 primes

3.1 $\zeta(p_n)$ is small

The main statement is that, if $\omega(p_n) > \zeta(p_n)$ for some p_n, the level $\zeta(p_n)$ has to be relatively small.
Theorem If $\omega(p_n) > \zeta(p_n)$, then $\zeta(p_n) \leq p_{n+1} - p_n$.

Proof Assume the statement to be false. Then we have a solution of the equation

$$2p_n - p_{n+1} = \omega(p_n)\zeta(p_n) \tag{10}$$

with $\omega(p_n) > \zeta(p_n) > p_{n+1} - p_n$ which would immediately imply that $\omega(p_n)$ cannot be the weight of p_n as $\zeta(p_n)$ does also fulfill the requirements and is smaller. Contradiction.

We can furthermore show that, if $\omega(p_n)$ is bigger than $\zeta(p_n)$, then it will be very big. In fact we will proof the following
Theorem If $\omega(p_n) > \zeta(p_n)$, then

$$\omega(p_n) \geq \frac{p_n}{p_{n+1}-p_n} - 1 \tag{11}$$

Proof We have

$$2p_n - p_{n+1} = \omega(p_n)\zeta(p_n) \tag{12}$$

and we do know that $\zeta(p_n) \leq p_{n+1} - p_n$. The theorem follows.

3.2 The primality of the weight in Zone2/3 primes

There are reasons to state the following

Conjecture If $\omega(p_n) > \zeta(p_n)$ then $\omega(p_n)$ is prime except for $p_6,\ p_{11},\ p_{30},\ p_{32}$ and p_{154}.

We can show the following weaker

Theorem Assume that $\omega(p_n) > \zeta(p_n)$. Then, if $\omega(p_n)$ is not a prime number, this implies that

$$(p_{n+1}-p_n)^3 \geq p_n - (p_{n+1}-p_n) \tag{13}$$

Proof We will first assume that $\omega(p_n)$ is not a prime number and then derive some statements about the prime factors of $\omega(p_n)$ which will lead us to the theorem stated above.
First of all we define

$$\omega(p_n) = \prod_{i=1}^{m} \alpha_i \tag{14}$$

Please notice that we are not using the canonic representation of primes but rather list them all (such as in $20 = 2 * 2 * 5$). We will also implicitly assume that $m \geq 2$ (otherwise $\omega(p_n)$ would be a prime anyway). Now we immediately find that all those prime factors have to be „small". Indeed, we have $\alpha_i \leq p_{n+1} - p_n$. If this was false for some special α_j we see that

1. α_j is a divisor of $p_n - (p_{n+1} - p_n)$ (because so is $\omega(p_n)$)
2. $\alpha_j > p_{n+1} - p_n$
3. $\alpha_j < \omega(p_n)$

Hence α_j does fulfill the necessary requirements for the weight and is actually smaller than $\omega(p_n)$. By definition the weight is the smallest number fulfilling the requirements and thus we have a contradiction.

Now, on the other hand, the prime factors cannot be too small. Indeed, we find

$$\forall 1 \leq i \leq m : \alpha_i \geq \frac{\omega(p_n)}{p_{n+1} - p_n} \tag{15}$$

To see this, we will again assume the equation to be false for some special α_j. If the equation was false, we could conclude that

$$p_{n+1} - p_n < \frac{\omega(p_n)}{\alpha_j} \tag{16}$$

But then we again find that

1. $\frac{\omega(p_n)}{\alpha_j}$ is a divisor of $p_n - (p_{n+1} - p_n)$
2. $\frac{\omega(p_n)}{\alpha_j} > p_{n+1} - p_n$
3. $\frac{\omega(p_n)}{\alpha_j} < \omega(p_n)$

Thus we found again a smaller number fulfilling the requirements while the weight is defined as the smallest such number and we have again reached a contradiction. Now, we found that for all α_i we have

$$\frac{\omega(p_n)}{p_{n+1} - p_n} \leq \alpha_i \leq p_{n+1} - p_n \tag{17}$$

Thus we immediately obtain that

$$\omega(p_n) \leq (p_{n+1} - p_n)^2 \tag{18}$$

Now, we can use the fact that $\zeta(p_n)$ is small and see that

$$p_n - (p_{n+1} - p_n) = \omega(p_n)\zeta(p_n) \leq (p_{n+1} - p_n)^2(p_{n+1} - p_n) = (p_{n+1} - p_n)^3 \tag{19}$$

3.3 How many Zone2/3 primes are there?

In this section we will assume the inequality $\Omega(n) > 0.6n$, which is yet nothing more than a mere conjecture, and show that there are not too few Zone2/3 primes.

Theorem Assuming $\Omega(n) > 0.6n$; if $\mathcal{B}$ is defined as the set of zone 2/3 primes among the set of prime numbers $\{p_5, p_6, p_7, ..., p_n\}$, then we have

$$|\mathcal{B}| \geq \frac{0.6n^2 - n\sqrt{p_n} + 4\sqrt{p_n}}{p_n - \sqrt{p_n}} \tag{20}$$

Proof We will first define two sets $\mathcal{A}$ and $\mathcal{B}$ with the properties $\mathcal{A} \cup \mathcal{B} = \{5, 6, ..., n\}$ and $\mathcal{A} \cap \mathcal{B} = \{\}$. the set $\mathcal{A}$ contains all those numbers i with the property $\omega(p_i) \leq \zeta(p_i)$ while in $\mathcal{B}$ we find those i for which $\omega(p_i) > \zeta(p_i)$. Then we apparently have $|\mathcal{A}| + |\mathcal{B}| = n - 4$. We will use this to split the sum into

$$\sum_{i=5}^{n} \omega(p_i) = \sum_{i \in \mathcal{A}} \omega(p_i) + \sum_{i \in \mathcal{B}} \omega(p_i) \tag{21}$$

The following inequality is apparent:

$$\sum_{i \in \mathcal{A}} \omega(p_i) \leq \sum_{i \in \mathcal{A}} \sqrt{p_i - (p_{i+1} - p_i)} \leq \sum_{i \in \mathcal{A}} \sqrt{p_i} \leq \sum_{i \in \mathcal{A}} \sqrt{p_n} = |\mathcal{A}| \sqrt{p_n} \tag{22}$$

Furthermore we also have

$$\sum_{i \in \mathcal{B}} \omega(p_i) \leq \sum_{i \in \mathcal{B}} p_i \leq |\mathcal{B}| \, p_n \tag{23}$$

Now, using the assumption $\Omega(n) > 0.6n$ we can derive, that

$$0.6n^2 \leq \sum_{i \in \mathcal{A}} \omega(p_i) + \sum_{i \in \mathcal{B}} \omega(p_i) \leq |\mathcal{A}| \sqrt{p_n} + |\mathcal{B}| \, p_n \tag{24}$$

Now consider that $|\mathcal{A}| = n - 4 - |\mathcal{B}|$ which leads us to

$$0.6n^2 \leq n\sqrt{p_n} - 4\sqrt{p_n} - |\mathcal{B}| \sqrt{p_n} + |\mathcal{B}| \, p_n = |\mathcal{B}| \, (p_n - \sqrt{p_n}) + n\sqrt{p_n} - 4\sqrt{p_n} \tag{25}$$

Now we can isolate $|\mathcal{B}|$ and effectively obtain that

$$\frac{0.6n^2 - n\sqrt{p_n} + 4\sqrt{p_n}}{p_n - \sqrt{p_n}} \leq |\mathcal{B}| \tag{26}$$

4 The average functions

4.1 Average of the weight: $\Omega(n)$

As it follows from the definition the value $\omega(p_n)$ is rather irregular. Therefore it does seem appropriate to study the average value, which we define as

$$\Omega(n) := \frac{1}{n}\sum_{i=5}^{n}\omega(p_i) \tag{27}$$

Theorem It is quite easy to show that $\Omega(n)$ is unbounded.

Proof

$$\Omega(n) = \frac{1}{n}\sum_{i=5}^{n}\omega(p_i) \geq \frac{1}{n}\sum_{i=5}^{n}(p_{i+1} - p_i + 1) = \frac{p_{n+1} - 11}{n} + \frac{n-4}{n} \tag{28}$$

Now, given the fact that $p_n > n\ln(n)$ due to the theorem of Rosser, we find that for $n \geq 15$

$$\Omega(n) > \frac{p_{n+1} - 11}{n} + \frac{n-4}{n} \geq \frac{p_n}{n} > \ln(n) \tag{29}$$

On the other hand some numerical experiments (for $1 \leq n \leq 5000$) indicate that for almost all n we find

$$\Omega(n) < 0.7n \tag{30}$$

4.2 Average of the level: $\xi(n)$

We define

$$\xi(n) = \frac{1}{n}\sum_{i=5}^{n}\zeta(p_i) \tag{31}$$

4.3 A bound on $\sum_{i=5}^{n}\omega(p_i) + \zeta(p_i)$

In this section we will attempt to show an inequality that gives us a lower bound for the growth of the sum of the average functions, by far better than $\ln(n)$. In fact we will attempt to prove the following

Theorem For all $n \geq 5$ we find

$$\sum_{i=5}^{n}\omega(p_i) + \zeta(p_i) \geq \frac{2}{3}n^{\frac{3}{2}} - \frac{16}{3} \tag{32}$$

Proof We will start with applying the easiest case of the AGM inequality. For positive numbers a and b it can be shown easily that $a + b \geq 2\sqrt{ab}$. Therefore we find that (using that $p_{n+1} \leq \frac{3p_n}{2}$)

$$\sum_{i=5}^{n}\omega(p_i) + \zeta(p_i) \geq 2\sum_{i=5}^{n}\sqrt{p_i - (p_{i+1} - p_i)} \geq 2\sum_{i=5}^{n}\sqrt{\frac{p_i}{2}} \tag{33}$$

Now we do know that $p_n > n\ln(n)$ and therefore it is easy to see that

$$2\sum_{i=5}^{n}\sqrt{\frac{p_i}{2}} = \sqrt{2}\sum_{i=5}^{n}\sqrt{p_i} \geq \sqrt{2}\sum_{i=5}^{n}\sqrt{n\ln(n)} \tag{34}$$

Now it should prove rather difficult to evaluate this sum, which is why we are using integrals now. $\sqrt{x\ln(x)}$ is a monotone increasing function, which immediately leads us to

$$\sqrt{n\ln(n)} \geq \int_{n-1}^{n}\sqrt{x\ln(x)}\ dx \tag{35}$$

Applying this inequality several times and adding the integrals gives us

$$\sqrt{2}\sum_{i=5}^{n}\sqrt{n\ln(n)} \geq \sqrt{2}\int_{4}^{n}\sqrt{x\ln(x)}\ dx \tag{36}$$

This integral can be evaluated exactly (Mathematica, for example, can do this) but in the result we find some nasty functions. Hence we will try to find a lower bound for that integral. Omitting the logarithm we get an easy integral, which we can evaluate without any problems.

$$\int_{4}^{n}\sqrt{x\ln(x)}\ dx \geq \int_{4}^{n}\sqrt{x}\ dx = \frac{2}{3}n^{\frac{3}{2}} - \frac{16}{3} \tag{37}$$

Hence we can finally say with certainty that

$$\sum_{i=5}^{n}\omega(p_i) + \zeta(p_i) \geq \frac{2}{3}n^{\frac{3}{2}} - \frac{16}{3} \tag{38}$$

This immediately leads us to

$$\sum_{i=5}^{n}\omega(p_i) + \zeta(p_i) = n(\Omega(n) + \xi(n)) \geq \frac{2}{3}n^{\frac{3}{2}} - \frac{16}{3} \tag{39}$$